© 2022 Ascham School Year 12 Mathematics Extension 1 Trial Examination

Name:___________________________

NESA number:____________________

Teacher: JH MN MA GS

ASCHAM SCHOOL

2022 **YEAR 12 TRIAL EXAMINATION**

Mathematics Extension 1

Friday 22nd July 2022

General Instructions

- Reading time – 10 minutes
- Working time – 2 hours
- Write using black non-erasable pen
- Calculators approved by NESA may be used
- A reference sheet is provided

Total marks: 70

Section I - 10 marks (pages 2 – 5)

- Use the multiple-choice answer sheet for Questions 1-10.
- Allow about 15 minutes for this section

Section II - 60 marks (pages 6 – 11)

- Attempt Questions 11–14, each worth 15 marks
- Allow about 1 hour and 45 minutes for this section
- Show relevant mathematical reasoning and/ or calculations for questions in this section
- Start a new booklet for each question

RESULT:

____/70 ________%

Section I

10 marks
Attempt questions 1 - 10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1 - 10

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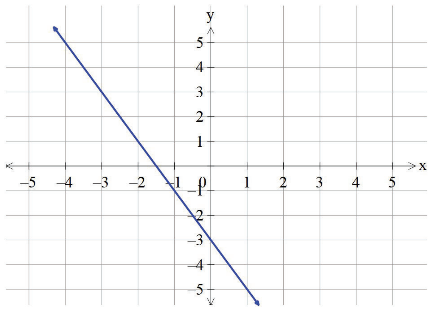
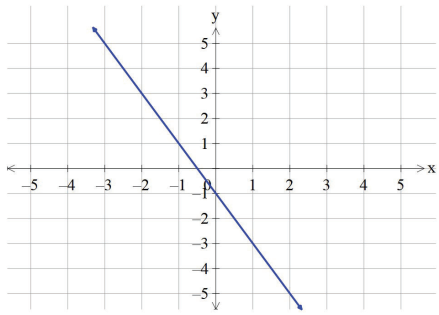
1. What is the graph of the function which has parametric equation $x = -\frac{p}{2} - 1, y = p + 1$?

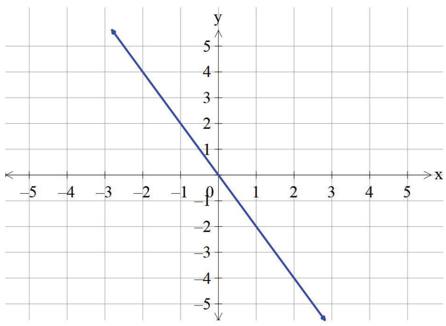
A.



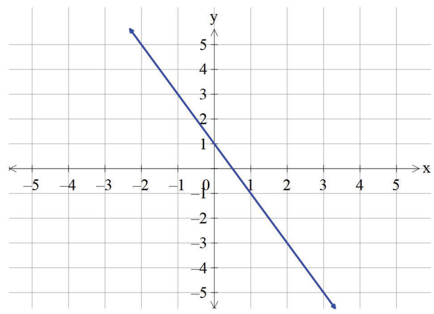
B.



C.



D.



2. Consider the two non-zero vectors, $\underset{\sim}{u}$ and $\underset{\sim}{v}$, with the following properties:

- $\underset{\sim}{v} = k\underset{\sim}{u}$, where k is a constant and
- $\underset{\sim}{u} \cdot \underset{\sim}{v} < 0$

Which of the following statements is true?

A. $\underset{\sim}{u} \perp \underset{\sim}{v}$

B. $\underset{\sim}{u} \parallel \underset{\sim}{v}$ in the same direction

C. $\underset{\sim}{u} \parallel \underset{\sim}{v}$ but in the opposite direction

D. The angle between the vectors $\underset{\sim}{u}$ and $\underset{\sim}{v}$ is acute

3. Which of the following is equivalent to $\int \frac{1-2x}{\sqrt{2x+1}}dx$, given the substitution $x = \frac{1}{2}(u-1)$?

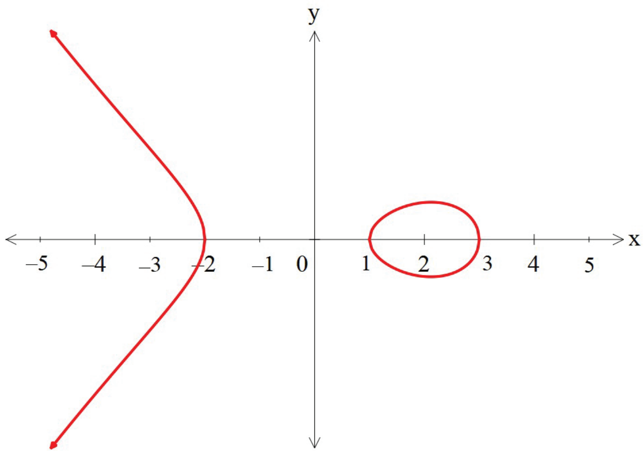
A. $\int \frac{2-u}{\sqrt{u}}du$

B. $2\int \frac{2-u}{\sqrt{u}}du$

C. $-\frac{1}{2}\int \frac{u}{\sqrt{u}}du$

D. $\int \frac{2-u}{2\sqrt{u}}du$

4. Given the graph of $y^2 = f(x)$, which of the following is the possible equation of $y = f(x)$?



A. $y=-(x+2)(x-1)(x-3)$

B. $y=(x+2)(x-1)(x-3)$

C. $y=-(x-3)(x-1)(x+2)^2$

D. $y=(3-x)(x+1)(x+2)$

5. Which of the following is equal to $\cos^2\left(\frac{\pi}{4}-x\right)-\sin^2\left(\frac{\pi}{4}-x\right)$?

A. 1

B. $\sin 2x$

C. $\cos 2x$

D. $\sin x$

6. What are the domain and range of $y = 2\cos^{-1}(\sin x)$?

A. $x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right],\ y \in \left[0, \frac{\pi}{2}\right]$

B. $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right),\ y \in [0, 2\pi]$

C. $x \in (-\infty, +\infty),\ y \in [0, 2\pi]$

D. $x \in (-\infty, +\infty),\ y \in [0, \pi]$

7. From a standard deck of cards, a card is drawn 20 times, with replacement. What is the probability that a red card is drawn exactly 15 times?

A. $\dfrac{{}^{20}C_5}{2^5}$

B. ${}^{20}C_{15}\left(\dfrac{1}{2}\right)^{15}$

C. $\dbinom{20}{5}\left(\dfrac{1}{2}\right)^{-20}$

D. $\dfrac{{}^{20}C_{15}}{2^{20}}$

8. What is the equation of $y = f^{-1}(x)$ if $y = x^2 - 2x - 2$ where $x \geq 1$?

A. $y = 1 \pm \sqrt{x+3}$

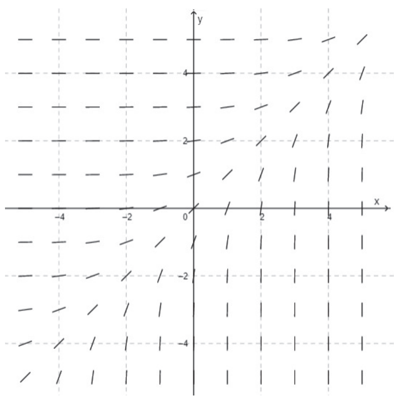
B. $y = 1 + \sqrt{x-3}$

C. $y = 1 - \sqrt{x+3}$

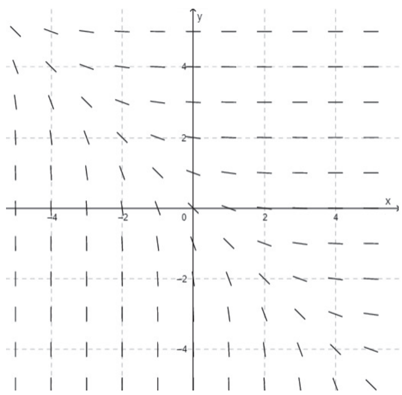
D. $y = 1 + \sqrt{x+3}$

9. Which of the following best represents the direction field for the differential equation $\dfrac{dy}{dx} = e^{x-y}$?

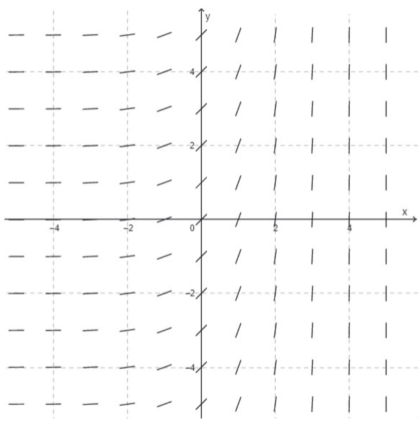
A.



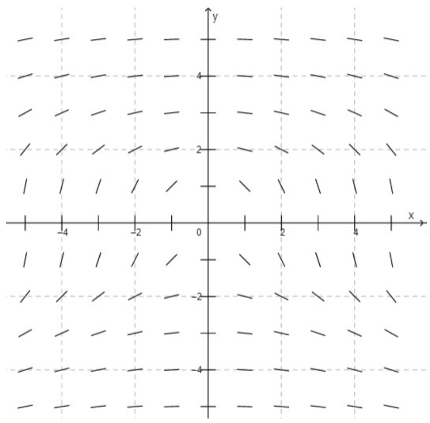
B.



C.



D.



10. A polynomial $P(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$ has real roots at x_1, x_2 and x_3, which are all **distinct** values of x. Given that $P'(x_1) = 0$, $P'(x_2) = P''(x_2) = 0$ and $P'(x_3) = P''(x_3) = P'''(x_3) = 0$, what is the lowest possible degree of $P(x)$?

A. $n = 9$

B. $n = 7$

C. $n = 6$

D. $n = 3$

End of Multiple-Choice Section

Section II

60 marks
Attempt questions 11 - 14
Allow about 1 hour and 45 minutes for this section

Answer each question in the appropriate writing booklet. Extra writing booklets are available.

For questions in Section II, your responses should include relevant mathematical reasoning and/ or calculations.

Question 11 (15 marks) Begin a new writing booklet

(a) Let $P(x) = 2x^3 + rx^2 + x - 3$ and $D(x) = x^2 + 6x - 4$ have the same remainder when divided by $x+1$. Find r. **[2]**

(b) Consider the vectors $\overrightarrow{OP} = -2\underset{\sim}{i} + \underset{\sim}{j}$ and $\overrightarrow{OQ} = 4\underset{\sim}{i} - 3\underset{\sim}{j}$.

(i) Find the vector $\underset{\sim}{u} = \overrightarrow{PQ}$. **[1]**

(ii) Given that $\underset{\sim}{v} = -\underset{\sim}{i} + 2\underset{\sim}{j}$, find $\underset{\sim}{u} \cdot \underset{\sim}{v}$. **[1]**

(iii) Find $proj_{\underset{\sim}{u}}\, \underset{\sim}{v}$. **[2]**

(c) What is the number of possible arrangements of the letters in the word LOGARITHMS, if 'G' is next to 'R'? **[2]**

(d) Solve $\sin 2x = \cos x$ for $0 \le x \le 180^\circ$. **[2]**

(e) (i) Write down the expansion of $(1-x)^6$. **[1]**

(ii) Hence, find the term in x^2 in the expansion $(2x-1)^2(1-x)^6$. **[2]**

(f) Evaluate exactly $\cos^{-1}\left(\sin \dfrac{4\pi}{3}\right)$. **[2]**

End of Question 11

Question 12 (15 marks) Begin a new writing booklet

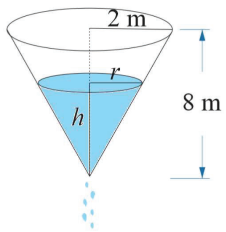
(a) Solve the differential equation $\frac{1}{1+x^2}\frac{dy}{dx} = \frac{x}{y}$ given that $y = -1$ when $x = 0$. **[3]**

(b) (i) Express $\cos x - \sqrt{3}\sin x$ in the form $R\cos(x+\alpha)$, where α is an acute angle. **[2]**

(ii) Hence, solve $\cos x - \sqrt{3}\sin x = 1$ for $-\pi \le x \le \pi$. **[2]**

(c) A box contains 16 red, 10 blue and 12 yellow balls. Use the pigeonhole principle to find the minimum number of balls to be drawn from the box to ensure 9 balls of same colour. **[2]**

(d) An inverted conical container is $8\,\text{m}$ deep and has the radius of the base $2\,\text{m}$. Water is leaking from the container at a constant rate of $\frac{dV}{dt} = 0.1\,\text{m}/\text{h}$, where V is the volume of the water in the container. Assume the container is full initially.



(i) Show that $V = \frac{\pi}{48}h^3$, where h is the height of the remaining water in the container. **[1]**

(ii) Hence, find the height of water in the container when $\frac{dh}{dt} = 0.02\,\text{m}/\text{h}$, correct to two decimal places. **[2]**

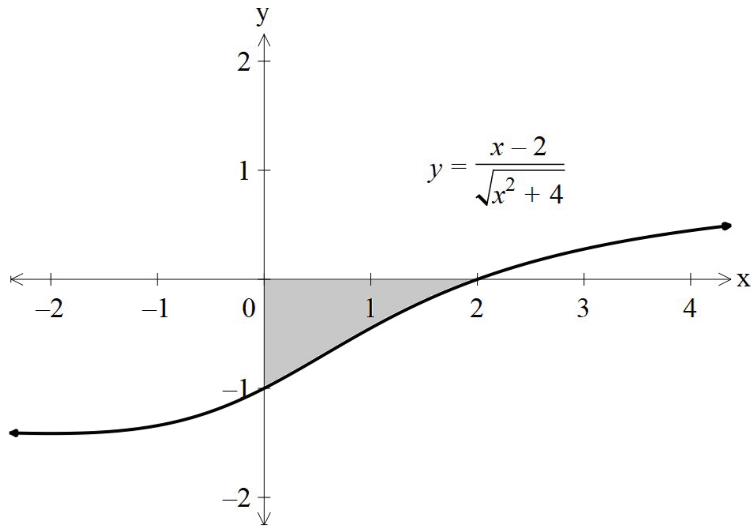
(e) (i) Show that $\frac{d}{dx}\left(x\tan^{-1}x\right) = \tan^{-1}x + \frac{x}{1+x^2}$. **[1]**

(ii) Hence, evaluate exactly $\int_0^{\sqrt{3}} \tan^{-1}x\,dx$. **[2]**

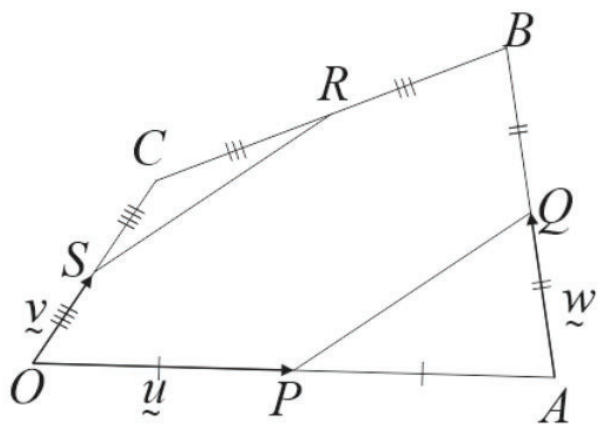
End of Question 12

Question 13 (15 marks) Begin a new writing booklet

(a) The region between the curve $y = \dfrac{x-2}{\sqrt{x^2+4}}$ and $y = 0$, from $x = 0$ to $x = 2$, has been rotated about the x - axis as per the diagram below. Find the volume of a solid of revolution formed, correct to four significant figures. **[2]**



(b) Consider the diagram below where $OABC$ is a quadrilateral and P, Q, R and S are the midpoints of the intervals OA, AB, BC and CO, respectively.



Let $\overrightarrow{OP} = \underset{\sim}{u}$, $\overrightarrow{OS} = \underset{\sim}{v}$ and $\overrightarrow{AQ} = \underset{\sim}{w}$. **[3]**

Prove that $\overrightarrow{PQ} = \overrightarrow{SR}$.

Question 13 continues on the next page…

(c) Use mathematical induction to prove that $3^{2n+1}+2^{n-1}$ is divisible by 7, for all positive integers n. **[3]**

(d) In the first half of 2022, 66% of all passengers departing from Sydney airport travelled internationally. A survey of 500 randomly selected people on the airport was conducted. **[3]**

Assuming that the sampling distribution of proportions $\hat{p}$ is approximately normally distributed, estimate the probability that the percentage of international travellers in the sample lies between 60% and 70%.

(e) (i) Show that $\dfrac{1-\cos 2x}{\sin 2x}+\dfrac{\sin 2x}{1+\cos 2x}=2\tan x$. **[2]**

(ii) Hence, show that $\tan\dfrac{\pi}{12}=2-\sqrt{3}$. **[2]**

End of Question 13

Question 14 (15 marks) Begin a new writing booklet

(a) After 100 snakes were released on an island on 1st January 2009 to control the population P_1 of 4000 cane toads, the number of cane toads started to decrease at a rate $\frac{dP_1}{dt} = -k(P_1 - 20)$ while the population of snakes, P_2, was changing according to $\frac{dP_2}{dt} = 0.0005P_2(500 - P_2)$.

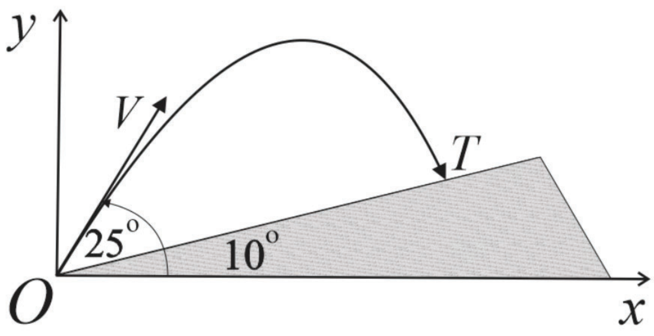
(i) Given that $P_1 = 20 + 3980e^{-kt}$ satisfies the differential equation $\frac{dP_1}{dt} = -k(P_1 - 20)$ and that there was 3119 cane toads left on the island after one year, show that $k = 0.25$ correct to two decimal places. **[1]**

(ii) Given that $\frac{1}{0.0005P_2(500 - P_2)} = 4\left(\frac{1}{P_2} + \frac{1}{500 - P_2}\right)$ show that the equation of the population P_2 of snakes is given by $P_2 = \frac{500}{1 + 4e^{-0.25t}}$. **[3]**

(iii) Assuming that $P_1 = 20 + 3980e^{-0.25t}$, in what year will the population of snakes exceed the population of cane toads? **[2]**

Question 14 continues on the next page…

(b) The diagram shows a road OT that makes an angle of 10° with the horizontal.



A projectile is fired from O at an angle of 25° to the horizontal, with initial velocity $V = 20\,\text{m/s}$. It hits a target at T.

Assume the acceleration due to gravity is $10\,\text{m/s}^2$.

(i) Find the time taken to hit the target. **[4]**

(ii) Find the distance OT. Give your answer correct to two decimal places. **[2]**

(c) Show that the function $y = e^{ax}\sin bx$ satisfies the equation $y'' - 2ay' + \left(a^2 + b^2\right)y = 0$. **[3]**

End of the Examination

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Probability table for the standard normal distribution

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990
3.1	0.9990	0.9991	0.9991	0.9991	0.9992	0.9992	0.9992	0.9992	0.9993	0.9993
3.2	0.9993	0.9993	0.9994	0.9994	0.9994	0.9994	0.9994	0.9995	0.9995	0.9995
3.3	0.9995	0.9995	0.9995	0.9996	0.9996	0.9996	0.9996	0.9996	0.9996	0.9997
3.4	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	0.9998

2022 Y12 EXT 1 TRIAL SOLUTIONS

SECTION I

[1] eg when $p=2$, $x=-2$, $y=3$

$p=0$, $x=-1$, $y=1$

Find the graph on which points $(-2,3)$ and $(-1,1)$ lie.

$\therefore$ (B)

Or eliminate p and use the gradient and y-int. to ident. graph

[2] $u+v=ku$ means u and v are parallel vectors

$u \cdot v < 0$ means angle between them is $180°$ $\therefore$ in opp. direct.

$\therefore$ (C)

[3] $\int \frac{1-2x}{\sqrt{2x+1}}\,dx$ $\quad x=\frac{1}{2}(u-1)$, $dx=\frac{1}{2}\,du$

$= \int \frac{1-2\times\frac{1}{2}(u-1)}{\sqrt{2\times\frac{1}{2}(u-1)+1}} \times \frac{1}{2}\,du$

$= \int \frac{1-u+1}{\sqrt{u-1+1}} \times \frac{1}{2}\,du$

$= \int \frac{2-u}{2\sqrt{u}}\,du$ $\quad \therefore$ (D)

[4] Check zeroes: $x=-2, 1, 3$

It has to be a cubic with neg. leading coeff. $\therefore$ (A)

[5] $\cos^2\left(\frac{\pi}{4}-x\right) - \sin^2\left(\frac{\pi}{4}-x\right)$

$= \cos\left[2\left(\frac{\pi}{4}-x\right)\right]$

$= \cos\left[\frac{\pi}{2}-2x\right]$

$= \cancel{\cos\frac{\pi}{2}\cos 2x} + \sin\frac{\pi}{2}\sin 2x$

$= \sin 2x$

$\therefore$ (B)

[6] $y=2\cos^{-1}(\sin x)$

Domain and range of $y=\cos^{-1}x$ $\quad D: [-1, 1]$

$R: [0, \pi]$

$\therefore$ Domain and range for $y=2\cos^{-1}(\sin x)$:

$D: -1 \le \sin x \le 1$

$\therefore x \in (-\infty, +\infty)$

as $\sin x$ is in that range for all x.

$R: [2\times 0, 2\times\pi] = [0, 2\pi]$

$\therefore$ (C)

[7] $Bin(20, 15) = {}^{20}C_{15} \times \left(\frac{1}{2}\right)^{15} \times \left(\frac{1}{2}\right)^{20}$

$= {}^{20}C_{15} \times \left(\frac{1}{2}\right)^{20}$

$= \frac{{}^{20}C_{15}}{2^{20}}$ $\quad \therefore$ (D)

[8] $y=x^2-2x-2$

$\therefore x=y^2-2y-2$

$x=y^2-2y+1-1-2$

$=(y-1)^2-3$

$\therefore (y-1)^2=x+3$ $\quad \therefore P(2,-2), P'(-2,2)$

$\therefore y=1\pm\sqrt{x+3}$ $\quad \therefore f^{-1}(x)=1+\sqrt{x+3}$ (D)

[9] DE is in the form $\frac{dy}{dx}=f(x)g(y)$

$\therefore$ gradients will change vertically and horizontally

$\therefore$ not C

$\frac{dy}{dx}$ is always positive

$\therefore$ not B and D $\quad \therefore$ (A)

[10] $P'(x_1)=0$ $\therefore$ double zero at $x=x_1$

$P''(x_2)=0$ $\therefore$ triple zero at $x=x_2$

$P'''(x_3)=0$ $\therefore$ quadr. zero at $x=x_3$

$\therefore n=2+3+4$

$=9$ $\quad \therefore$ (A)

SECTION II

QUESTION 11

a) $P(x) = 2x^3 + rx^2 + x - 3$

$D(x) = x^2 + 6x - 4$

$\therefore D(-1) = (-1)^2 + 6(-1) - 4$

$= -9$

$P(-1) = -2 + r - 1 - 3$

$= r - 6$

Since $D(-1) = P(-1)$

$r - 6 = -9$

$\therefore r = -3$ [2]

b) $\overrightarrow{OP} = -2\underset{\sim}{i} + \underset{\sim}{j}$ $\quad \overrightarrow{OQ} = 4\underset{\sim}{i} - 3\underset{\sim}{j}$

i) $\underset{\sim}{u} = \overrightarrow{PQ}$

$= \overrightarrow{PO} + \overrightarrow{OQ}$

$= -(\overrightarrow{OP}) + \overrightarrow{OQ}$

$= -(-2\underset{\sim}{i} + \underset{\sim}{j}) + 4\underset{\sim}{i} - 3\underset{\sim}{j}$

$= 2\underset{\sim}{i} - \underset{\sim}{j} + 4\underset{\sim}{i} - 3\underset{\sim}{j}$

$= 6\underset{\sim}{i} - 4\underset{\sim}{j}$ [1]

ii) $\underset{\sim}{u} \cdot \underset{\sim}{v} = (6\underset{\sim}{i} - 4\underset{\sim}{j}) \cdot (-\underset{\sim}{i} + 2\underset{\sim}{j})$

$= 6 \times (-1) + (-4) \times 2$

$= -14$ [1]

iii) $\text{proj}_{\underset{\sim}{u}} \underset{\sim}{v} = \dfrac{\underset{\sim}{u} \cdot \underset{\sim}{v}}{|\underset{\sim}{u}|^2} \underset{\sim}{u}$

$\therefore \text{proj}_{\underset{\sim}{u}} \underset{\sim}{v} = \dfrac{-14}{(\sqrt{6^2 + (-4)^2})^2} (6\underset{\sim}{i} - 4\underset{\sim}{j})$

$= \dfrac{-14}{52} (6\underset{\sim}{i} - 4\underset{\sim}{j})$

$= \dfrac{-84}{52} \underset{\sim}{i} + \dfrac{56}{52} \underset{\sim}{j}$

$= -\dfrac{21}{13} \underset{\sim}{i} + \dfrac{14}{13} \underset{\sim}{j}$ [2]

c) 10 letters

GR or RG _ _ _ _ _ _ _ _ [2]

$\therefore$ Number of arrang. $= 9! \times 2!$

$= 725760$

d) $\therefore 2\sin x \cos x = \cos x$

[2] $2\sin x \cos x - \cos x = 0$

$\cos x (2\sin x - 1) = 0$

$\therefore \cos x = 0$ or $\sin x = \frac{1}{2}$

$\therefore x = 90°$ $\quad \therefore x = 30°, 150°$

$\therefore x = 30°, 90°, 150°$

as $0° \le x \le 180°$

e)

i) $(1-x)^6 = (1 + (-x))^6$

[2] $= {}^6C_0 1^6 (-x)^0 + {}^6C_1 1^5 (-x)^1$

$+ {}^6C_2 1^4 (-x)^2 + {}^6C_3 1^3 (-x)^3$

$+ {}^6C_4 1^2 (-x)^4 + {}^6C_5 1^1 (-x)^5$

$+ {}^6C_6 1^0 (-x)^6$

$= 1 - 6x + 15x^2 - 20x^3 + 15x^4 - 6x^5 + x^6$

ii) $(2x-1)^2 = 4x^2 - 4x + 1$

$\therefore$ The term in x^2

$= 4x^2 \times 1 - 4x \times (-6x) + 1 \times 15x^2$

$= 43x^2$ [2]

f)

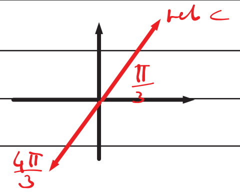
$\cos^{-1}(\sin \frac{4\pi}{3}) = \cos^{-1}(-\sin \frac{\pi}{3})$ *

$= \cos^{-1}(-\frac{\sqrt{3}}{2})$ **

$= \pi - \frac{\pi}{6}$

$= \frac{5\pi}{6}$ [2]

*



**

$0 \le \cos^{-1}(-\frac{\sqrt{3}}{2}) \le \pi$

$\therefore \cos^{-1}(-\frac{\sqrt{3}}{2})$ is in 2nd quadr

with rel $< = \frac{\pi}{6}$

QUESTION 12

a) $\frac{1}{1+x^2}\frac{dy}{dx} = \frac{x}{y}$, $x=0$ $y=-1$

[3] $\therefore y\,dy = x(1+x^2)dx$ $/\int$

$\int y\,dy = \int (x+x^3)dx$

$\therefore \frac{y^2}{2} = \frac{x^2}{2}+\frac{x^4}{4}+C$

$\therefore y^2 = x^2+\frac{x^4}{2}+C_1$

$\therefore y = \pm\sqrt{x^2+\frac{x^4}{2}+C_1}$

Given that $y=-1$ when $x=0$

$y = -\sqrt{x^2+\frac{x^4}{2}+C_1}$

$\therefore -1 = -\sqrt{0+\frac{0}{2}+C_1}$ $\therefore C_1 = 1$

$\therefore y = -\sqrt{x^2+\frac{x^4}{2}+1}$

b)

i) $\cos x - \sqrt3 \sin x = R\cos(x+\alpha)$

$\cos x-\sqrt3\sin x = R\cos x\cos\alpha - R\sin x\sin\alpha$

$\therefore R\cos\alpha = 1$ $/^2$

$+$ $R\sin\alpha = \sqrt3$ $/^2$

$R^2(\overset{1}{\cancel{\cos^2\alpha+\sin^2\alpha}}) = 4$

$R^2 = 4$

$\therefore R = 2$ as $R>0$

Since $\cos\alpha = \frac12 > 0$ and

$\sin\alpha = \frac{\sqrt3}{2} > 0$

α is in 1st quadr and

$\alpha = \frac{\pi}{3}$

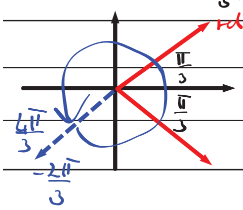
$\therefore \cos x - \sqrt3\sin x = R(x+\frac{\pi}{3})$ [2]

ii) $\cos x-\sqrt3\sin x = 1$ $\quad -\pi\le x\le\pi$

$\therefore 2\cos(x+\frac\pi3)=1$

$\therefore \cos(x+\frac\pi3)=\frac12$ $\quad -\frac{2\pi}{3}\le x+\frac\pi3\le\frac{4\pi}{3}$

Since $\cos(x+\frac\pi3)>0$ $\quad$ rel $\angle = \frac\pi3$



$x+\frac\pi3 = -\frac\pi3, \frac\pi3$

$\therefore x = -\frac{2\pi}{3}, 0$

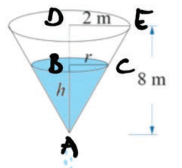
[2]

c) No. of colours (pigeonholes) = 3

$\therefore$ Min. no. of balls $= 3\times 8+1$
$= 25$ [2]

(eg. 8red+8blue+8yellow+1(r or b or y))

d)



i) $\Delta ABC \parallel\!| \Delta ADE$ $\quad \therefore V=\frac13\pi r^2 h$

(Equiangular) $\quad = \frac13\pi\left(\frac h4\right)^2 h$

$\therefore \frac2r = \frac8h$ $\quad = \frac{\pi}{48}h^3$

$\therefore r = \frac{2h}{8}$ $\quad$ [1]

$= \frac h4$

ii) $\therefore \frac{dV}{dt} = \frac{dV}{dh}\times\frac{dh}{dt}$

$0.1 = \frac{d}{dh}\left(\frac{\pi}{48}h^3\right)\times 0.02$

$= \frac{\pi}{16}h^2\times 0.02$

$\therefore \frac{\pi}{16}h^2 = 5$

$h^2 = \frac{80}{\pi}$

$\therefore h = 5.0462\ldots$

$\doteqdot 5.05$ m [2]

e) i) $y = \underbrace{x}_{u}\underbrace{\tan^{-1}x}_{v}$

Let $u = x$, $v = \tan^{-1}x$

Then $u' = 1$, $v' = \frac{1}{1+x^2}$

$\frac{d}{dx}(\underbrace{x}_{u}\,\underbrace{\tan^{-1}x}_{v}) = vu'+uv'$

$= \tan^{-1}x + \frac{x}{1+x^2}$ [1]

ii) $\tan^{-1}x = \frac{d}{dx}(x\tan^{-1}x) - \frac{x}{1+x^2}$ $/\int$

$\therefore \int\tan^{-1}x\,dx = x\tan^{-1}x - \int\frac{x}{\underset{f'(x)}{1+x^2}}dx$

$= x\tan^{-1}x - \frac12\int\frac{2x}{\underset{f(x)}{1+x^2}}dx$

$= x\tan^{-1}x - \frac12\ln|1+x^2| + C$

$= \left[x\tan^{-1}x\right]_0^{\sqrt3} - \frac12\left[\ln|1+x^2|\right]_0^{\sqrt3}$

$= \left[\sqrt3\times\frac\pi3 - 0\right] - \frac12\left[\ln4-\ln1\right]$

$= \frac{\sqrt3\pi}{3} - \frac{\ln4}{2}$

$= \frac{2\sqrt3\pi-3\ln4}{6}$ [2]

QUESTION 13

a) [2]

$$V=\pi\int_0^2 y^2\,dx$$

$$=\pi\int_0^2 \frac{(x-2)^2}{(\sqrt{x^2+4})^2}\,dx$$

$$=\pi\int_0^2 \frac{x^2-4x+4}{x^2+4}\,dx$$

$$=\pi\int_0^2\left(\frac{x^2+4}{x^2+4}-\frac{4x}{x^2+4}\right)dx$$

$$=\pi\int_0^2\left(1-2\frac{2x}{x^2+4}\right)dx$$

$$=\pi\left[x-2\ln|x^2+4|\right]_0^2$$

$$=\pi\left[2-2\ln 8-0+2\ln 4\right]$$

$$=1.92801\ldots$$

$$=1.928 \ (4\text{ s.f.})$$

b) [3]

$$\overrightarrow{PQ}=\overrightarrow{PA}+\overrightarrow{AQ}$$

$$=\tfrac{1}{2}\overrightarrow{OA}+\overrightarrow{AQ}$$

$$=\overrightarrow{OP}+\overrightarrow{AQ}$$

$$=\underset{\sim}{u}+\underset{\sim}{w}$$

$$\overrightarrow{SR}=\overrightarrow{SC}+\overrightarrow{CR}$$

$$=\tfrac{1}{2}\overrightarrow{OC}+\tfrac{1}{2}\overrightarrow{CB}$$

$$=\overrightarrow{OS}+\tfrac{1}{2}\left(\overrightarrow{CO}+\overrightarrow{OA}+\overrightarrow{AB}\right)$$

$$=\underset{\sim}{v}+\tfrac{1}{2}\left(-2\underset{\sim}{v}+2\underset{\sim}{u}+2\underset{\sim}{w}\right)$$

$$=\cancel{\underset{\sim}{v}}-\cancel{\underset{\sim}{v}}+\underset{\sim}{u}+\underset{\sim}{w}$$

$$=\underset{\sim}{u}+\underset{\sim}{w}$$

$$\therefore\ \overrightarrow{PQ}=\overrightarrow{SR}$$

c) Let $P(n)$ be the proposition
[3] that $3^{2n+1}+2^{n-1}$ is divisible by 7
for all integers $n>0$.

Prove that $P(1)$ is true.

$\therefore$ For $n=1$, $3^{2n+1}+2^{n-1}=3^3+2^0$
$=28$

Since 28 is divisible by 7,
$P(1)$ is true.

Assume that $P(k)$ is true. That is,

$3^{2k+1}+2^{k-1}=7p$, p is an integer

$\therefore\ 2^{k-1}=7p-3^{2k+1}$

$\frac{2^k}{2}=7p-3^{2k+1}$

$\therefore\ 2^k=2(7p-3^{2k+1})$ *

Prove that $P(k+1)$ is true. That is

$3^{2(k+1)+1}+2^{(k+1)-1}=7q$, q integer

$\therefore$ LHS $=3^{2(k+1)+1}+2^{(k+1)-1}$

$=3^{2k+3}+2^k$

$=3^{2k+3}+2\times 7p-2\times 3^{2k+1}$

$=9\times 3^{2k+1}+2\times 7p-2\times 3^{2k+1}$

$=7\times 3^{2k+1}+7\times 2p$

$=7(3^{2k+1}+2p)$

$=7q$, q is an integer

$=$ RHS

$\therefore$ If $P(k)$ is true, $P(k+1)$ is also true

$\therefore$ The proposition $P(n)$ is true
by the principle of Math. Ind.
for all $n>0$.

d) Population : $p=0.66$ $q=1-p$
$=0.34$

[3] Sample : $\hat{p}$, $\mu_{\hat{p}}=p=0.66$, $n=500$

$P(0.6\le \hat{p}\le 0.7)$

$$=P\left(\frac{0.6-0.66}{\sqrt{\frac{0.66\times 0.34}{500}}}\le z\le \frac{0.7-0.66}{\sqrt{\frac{0.66\times 0.34}{500}}}\right)$$

$=P(-2.8322\ldots\le z\le 1.88813\ldots)$

$=P(-2.83\le z\le 1.89)$

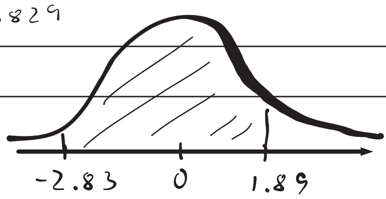
$=P(z\le 1.89)-P(z\le -2.83)$

$=P(z\le 1.89)-P(z\ge 2.83)$

$=P(z\le 1.89)-1+P(z\le 2.83)$ Table

$=0.97062-1+0.99767$

$=0.96829$



e)

i) LHS $= \frac{1-\cos 2x}{\sin 2x} + \frac{\sin 2x}{1+\cos 2x}$

[2]

$= \frac{1-\cos^2 x+\sin^2 x}{2\sin x\cos x} + \frac{2\sin x\cos x}{1+\cos^2 x-\sin^2 x}$

$= \frac{\cancel{2}\sin^2 x}{\cancel{2}\cancel{\sin x}\cos x} + \frac{\cancel{2}\sin x\cancel{\cos x}}{\cancel{2}\cos^{\cancel{2}} x}$

$= \frac{\sin x}{\cos x} + \frac{\sin x}{\cos x}$

$= 2\tan x$

$=$ RHS

ii) $2\tan\frac{\pi}{12} = \frac{1-\cos(2\times\frac{\pi}{12})}{\sin(2\times\frac{\pi}{12})} + \frac{\sin(2\times\frac{\pi}{12})}{1+\cos(2\times\frac{\pi}{12})}$

[2]

$= \frac{1-\cos\frac{\pi}{6}}{\sin\frac{\pi}{6}} + \frac{\sin\frac{\pi}{6}}{1+\cos\frac{\pi}{6}}$

$= \frac{1-\frac{\sqrt{3}}{2}}{\frac{1}{2}} + \frac{\frac{1}{2}}{1+\frac{\sqrt{3}}{2}}$

$= 2-\sqrt{3} + \frac{1}{2+\sqrt{3}} \times \frac{2-\sqrt{3}}{2-\sqrt{3}}$

$= 2-\sqrt{3} + \frac{2-\sqrt{3}}{4-3}$

$= 4-2\sqrt{3}$

$\therefore \tan\frac{\pi}{12} = \frac{4-2\sqrt{3}}{2}$

$= 2-\sqrt{3}$

QUESTION 14

i) [1]

$P_1 = 20 + 3980e^{-kt}$

When $t = 1$, $P_1 = 3119$

$\therefore 3119 = 20 + 3980e^{-k}$

$\therefore e^{-k} = \dfrac{3119-20}{3980}$ / ln

$\therefore -k = \ln\left(\dfrac{3099}{3980}\right)$

$\therefore k \doteqdot 0.25$

$\therefore P_1 = 20 + 3980e^{-0.25t}$

ii) [3]

$\dfrac{dP_2}{dt} = 0.0005P_2(500-P_2)$

$\therefore \dfrac{dt}{dP_2} = \dfrac{1}{0.0005P_2(500-P_2)}$

$\therefore \dfrac{dt}{dP_2} = 4\left(\dfrac{1}{P_2} + \dfrac{1}{500-P_2}\right)$ / $\int$

$\therefore t = 4(\ln|P_2| - \ln|500-P_2|)$

$= 4\ln\left|\dfrac{P_2}{500-P_2}\right| + C$

$\therefore t - C = 4\ln\left|\dfrac{P_2}{500-P_2}\right|$

$\therefore \ln\left|\dfrac{P_2}{500-P_2}\right| = \dfrac{1}{4}(t-C)$

$= \dfrac{t}{4} - \dfrac{C}{4}$

$\therefore \left|\dfrac{P_2}{500-P_2}\right| = e^{0.25t} \times e^{0.25C}$

$\therefore \dfrac{P_2}{500-P_2} = \pm e^{0.25C} \times e^{0.25t}$

$= Ae^{0.25t}$, $A = \pm e^{0.25C}$

Finding A: Subst $t = 0$, $P_2 = 100$

$\therefore \dfrac{100}{400} = A \quad \therefore A = \dfrac{1}{4}$

$\therefore \dfrac{P_2}{500-P_2} = \dfrac{e^{0.25t}}{4}$

$\therefore 4P_2 = 500e^{0.25t} - P_2e^{0.25t}$

$\therefore P_2(4 + e^{0.25t}) = \dfrac{500}{e^{-0.25t}}$

$\therefore P_2 = \dfrac{500}{\frac{1}{e^{0.25t}} \times (4 + e^{0.25t})}$

$= \dfrac{500}{1 + 4e^{-0.25t}}$

iii) [2]

Find the time when $P_2 = P_1$

$\therefore \dfrac{500}{1+4e^{-0.25t}} = 20 + 3980e^{-0.25t}$

$500 = 20 + 80e^{-0.25t} + 3980e^{-0.25t} + 15920(e^{-0.25t})^2$

Forming quadratic:

$\therefore 15920(e^{-0.25t})^2 + 4060e^{-0.25t} - 480 = 0$

$e^{-0.25t} = \dfrac{-4060 \pm \sqrt{\ldots}}{2 \times 15920}$

$\therefore e^{-0.25t} = \dfrac{-4060 + 6859.300\ldots}{31840}$

since $e^{-0.25} > 0$

$\therefore -0.25t = \ln(0.087917\ldots)$

$\therefore t = 9.7254\ldots$ years

$\therefore P_2$ will exceed P_1 some time in 2018.

Students can also solve $P_2 > P_1$.

b)

i) Let $\underset{\sim}{a} = -g\underset{\sim}{j}$ Then: $\ddot{x} = 0$ and $\ddot{y} = -g$; $V = 20$ m/s

[4]

Finding velocity:

$$\underset{\sim}{v} = \int \underset{\sim}{a}\, dt$$
$$= \int -10\underset{\sim}{j}\, dt$$
$$= -10t\underset{\sim}{j} + \underset{\sim}{C}$$

Finding $\underset{\sim}{C}$:

$\underset{\sim}{v} = -10t\underset{\sim}{j} + \underset{\sim}{C}$ ∴ when $t = 0$, $\underset{\sim}{v} = \underset{\sim}{C}$

Also, initial velocity is $\underset{\sim}{v} = 20\cos\theta\underset{\sim}{i} + 20\sin\theta\underset{\sim}{j}$

∴ $\underset{\sim}{C} = v\cos\theta\underset{\sim}{i} + v\sin\theta\underset{\sim}{j}$

∴ $\underset{\sim}{v} = -10t\underset{\sim}{j} + 20\cos\theta\underset{\sim}{i} + 20\sin\theta\underset{\sim}{j}$

Now, $\underset{\sim}{s} = \int \underset{\sim}{v}\, dt$

$$= \int \left[20\cos\theta\underset{\sim}{i} + (-10t + 20\sin\theta)\underset{\sim}{j}\right] dt$$
$$= 20t\cos\theta\underset{\sim}{i} + \left(\frac{-10t^2}{2} + 20t\sin\theta\right)\underset{\sim}{j} + \underset{\sim}{D}$$

Initially when $t = 0$, $|\underset{\sim}{s}| = 0$ ∴ $\underset{\sim}{s} = 0\underset{\sim}{i} + 0\underset{\sim}{j}$

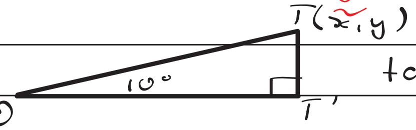
∴ $0 = 20\times0\times\cos\theta\underset{\sim}{i} + (-5\times0 + 20\times0\times\sin\theta)\underset{\sim}{j} + \underset{\sim}{D}$

∴ $\underset{\sim}{D} = \underset{\sim}{0}$

$$\underset{\sim}{s} = 20t\cos\theta\underset{\sim}{i} + (-5t^2 + 20t\sin\theta)\underset{\sim}{j}$$

∴ $\underset{\sim}{s} = 20t\cos25^\circ\underset{\sim}{i} + (-5t^2 + 20t\sin25^\circ)\underset{\sim}{j}$

Given the diagram:



$\tan10^\circ = \frac{y}{x}$

∴ $\tan10^\circ = \frac{-5t^2 + 20t\sin25^\circ}{20t\cos25^\circ}$

$$= \frac{\cancel{5t}(4\sin25^\circ - t)}{4\,\cancel{20t}\cos25^\circ}$$
$$= \frac{4\sin25^\circ - t}{4\cos25^\circ}$$

∴ $t = 4\sin25^\circ - 4\tan10^\circ\cos25^\circ$

$= 1.05124\ldots$ seconds

ii) $\cos10^\circ = \frac{OT'}{OT}$

[2]

$$= \frac{x}{OT}$$

∴ $OT = \frac{x}{\cos10^\circ}$

∴ $OT = \frac{20\times1.05124\ldots\times\cos25^\circ}{\cos10^\circ}$

$= 19.34889\ldots$

$\doteqdot 19.35$ m

c) $y = \underbrace{e^{ax}}_{u} \underbrace{\sin bx}_{v}$ Let $u = e^{ax}$, $v = \sin bx$

[3]   Then, $u' = ae^{ax}$, $v' = b\cos bx$

$\therefore y' = uv' + vu'$

$= be^{ax}\cos bx + ae^{ax}\sin bx$

$= \underbrace{e^{ax}}_{u}\underbrace{(b\cos bx + a\sin bx)}_{v}$

Let $u = e^{ax}$, $v = b\cos bx + a\sin bx$

Then $u' = ae^{ax}$, $v' = -b^2\sin bx + ab\cos bx$

$\therefore y'' = uv' + vu'$

$= e^{ax}(-b^2\sin bx + ab\cos bx) + ae^{ax}(b\cos bx + a\sin bx)$

To prove: $y'' - 2ay' + (a^2+b^2)y = 0$

$\therefore LHS = y'' - 2ay' + (a^2+b^2)y$

$= e^{ax}(-b^2\sin bx + ab\cos bx + ab\cos bx + a^2\sin bx)$

$- e^{ax}(2ab\cos bx + 2a^2\sin bx)$

$+ e^{ax}(a^2\sin bx + b^2\sin bx)$

$= e^{ax}(\cancel{-b^2\sin bx} + \cancel{2ab\cos bx} + \cancel{a^2\sin bx}$

$\cancel{- 2ab\cos bx} \cancel{- 2a^2\sin bx} + \cancel{a^2\sin bx} + \cancel{b^2\sin bx})$

$= e^{ax} \times 0$

$= 0$

$= RHS$